Foul Or Not?

Strategy is certainly crucial to any sport or any situation where we have a confrontation or conflict.

- In this section, we will see how probability and expected value can help us make better decisions on strategy, for example;
- Should a team foul or not in order to keep our lead at the end of a basketball game?
- How often should a football team run the ball and how often should they pass it?
- How often should a soccer goalie dive to the left and how often to the right?

Tree Diagrams

One very useful tool in calculating probability or making decisions on strategy is a a tree diagram. A tree diagram is very useful for choosing a strategy in an alternating move game such as chess.

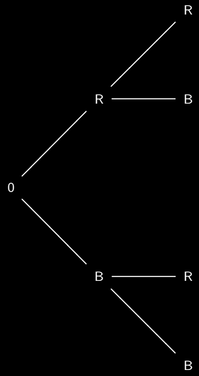
It is also especially useful for calculating probability when:

- there are sequential steps in an experiment or
- or when we carry out repeated trials of the same experiment,
- or if there are a number of stages of classification for objects sampled

Example: Tree Diagram

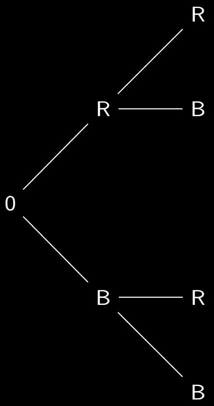
Given an Urn containing 6 red marbles and 4 blue marbles, I draw a marble at random from the urn and then, without replacing the first marble, I draw a second marble from the urn. What is the probability that both marbles are red?

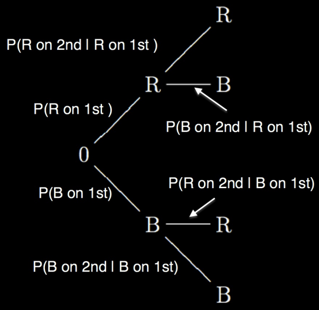
- We can draw a tree diagram to represent the possible outcomes of the above experiment and label it with the appropriate conditional probabilities as shown (where 1st denotes the first draw and 2nd denotes the second draw):



Example: Tree Diagram

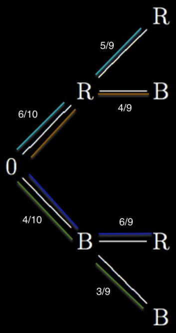
Given an Urn containing 6 red marbles and 4 blue marbles, I draw a marble at random from the urn and then, without replacing the first marble, I draw a second marble from the urn. Fill in the probabilities on the tree diagram below.

- 



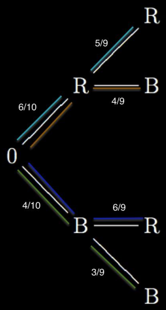
Example: Tree Diagram

Given an Urn containing 6 red marbles and 4 blue marbles, I draw a marble at random from the urn and then, without replacing the first marble, I draw a second marble from the urn. Note that each path on the tree diagram represents one outcome in the sample space. To find the probability of an outcome we multiply probabilities along the paths;
$P(RR) = P(R \text{ on 1st})P(R \text{ on 2nd}|R \text{ on 1st})$ etc.... (Here we are using the formula $P(A \cap B) = P(A)P(B|A)$.)



Outcome	Probability
RR(light blue path)	
RB(brown path)	
BR(dark blue path)	
BB(green path)	

Example: Tree Diagram



Outcome	Probability
RR(light blue path)	$\frac{6}{10}\frac{5}{9} = \frac{1}{3}$
RB(brown path)	$\frac{6}{10}\frac{4}{9} = \frac{24}{90}$
BR(dark blue path)	$\frac{4}{10}\frac{6}{9} = \frac{24}{90}$
BB(green path)	$\frac{4}{10}\frac{3}{9} = \frac{12}{90}$

- The event that both marbles are red corresponds to a single path RR, thus the probability is the product of the probabilities along the path : $P(RR) = \frac{6}{10}\frac{5}{9} = \frac{1}{3}$.
- The event that the second marble is blue is more complex; there are two paths in this event.
- $P(\text{second marble blue}) = P(RB) + P((BB) = \frac{6}{10}\frac{4}{9} + \frac{4}{10}\frac{3}{9} = \frac{24}{90} + \frac{12}{90} = \frac{36}{90}$.

Tree Diagrams: Rules for Construction

- The branches emanating from each point (that is branches on the immediate right) must represent all possible outcomes in the next stage of classification or in the next experiment.
- The sum of the probabilities on this bunch of branches adds to 1
- We label the paths with appropriate conditional probabilities.

Tree Diagrams: Rules for Calculation

- Each path corresponds to some outcome
- The probability of that outcome is the product of the probabilities along the path
- To calculate the probability of an event E, collect all paths in the event E, calculate the probability for each such path and then add the probabilities of those paths.

Tree Diagrams: Choosing a Strategy

Tree diagrams are very helpful when choosing strategies which maximize the probability of winning or expected gains.

- In some of these situations, the only opponent is chance itself
- and in others we have an opponent with players making alternate decisions as in chess.
- We will first give an example where chance is the opponent and then we will look at an example of an alternate move game.

Endgame Basketball: Choosing a Strategy

Our first example comes from a paper by Coach Bill Fenlon, head basketball coach at DePaw University. The paper is called: *Up Theee, To Foul or Not To Foul* and is available on the internet. In this paper Coach Fenlon looks at the following scenario:

- there is very little time left on the clock,
- your team has a three point lead and
- the opposing team has possession of the ball.
- If the opposing team shoots for three points and makes the shot, the game will go into overtime.

Endgame Basketball: Choosing a Strategy

If your team fouls, the opposing team will be granted free throws, the number of which depends on the type of foul:

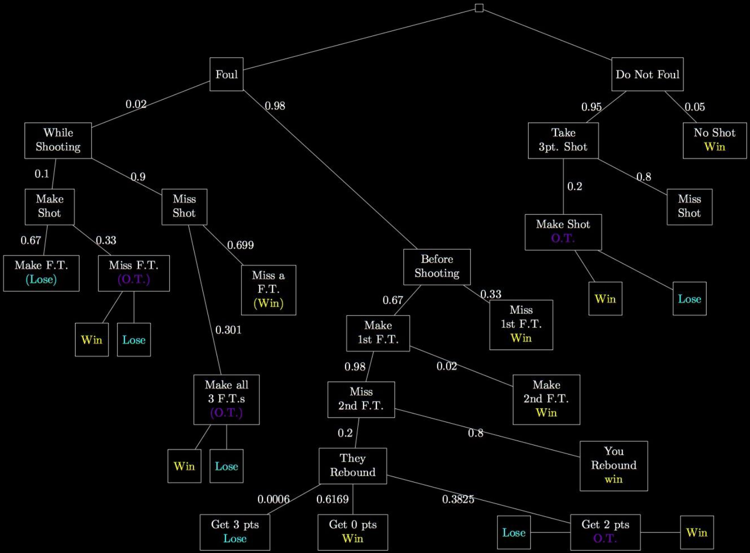
- If you foul as the opponent is shooting and the shot is made, your opponent will be granted one free throw.
- If you foul as the opponent is shooting and the shot is missed, your opponent will be granted three free throws.
- If you foul before your opponent shoots, your opponent will be granted two free throws.
- Each free throw made is worth one point. If the teams draw, the game will go into overtime(O.T.).

Endgame Basketball: Choosing a Strategy

Coach Fenlon gives the following estimates of probability based on national statistics (these probabilities may vary deepening on the level at which the team is playing or the strength of defense or offense of the teams involved).

- Coach Fenlon estimates that the probability that the other team will make a free throw (if that is their intention) in this situation is 0.67.
- He estimates that with practice, you can reduce the probability of fouling while the opponent is shooting to 0.02.
- He points out that if you foul before the opponent takes the 3 point shot and they are granted 2 free throws, they will attempt to miss the second free throw and go for a rebound shot. In this case, he estimates that they will accidentally make the second shot 2% of the time.
- If you do not foul, he estimates that the opposing team will not get a chance to make a 3 point shot 5% of the time and if they attempt the shot, they will be successful 20% of the time.
- Coach Fenlon also gives other detailed estimates which we have represented on a less detailed tree diagram on the next slide.

Endgame Basketball: Choosing a Strategy



Endgame Basketball: Choosing a Strategy

Using the tree diagram, let's fill in the following probability distribution for the results at the end of regulation time (normal duration of the game) for both strategies:

Strategy Foul

Outcome	Probability
Win (W)	
Loss(L)	
Overtime(O.T.)	

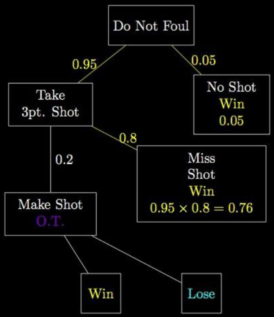
Strategy: Do Not Foul

Outcome	Probability
Win (W)	
Loss(L)	
Overtime(O.T.)	

- To do this, we must identify the paths in the diagram corresponding to each event (foul and win, foul and lose etc...), multiply probabilities along each path and then add the probabilities for all paths corresponding to the event.

For Strategy "Do Not Foul"

For the Strategy "Do Not Foul", we calculate the probability of a Win in Regulation Time by identifying each path ending with a win in regulation time (shown in yellow in the diagram on left).



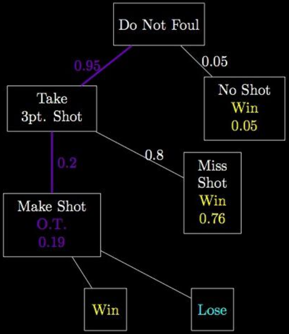
Strategy: Do Not Foul

Outcome	Probability
Win (W)	0.81
Loss(L)	
Overtime(O.T.)	

- We multiply the probabilities along each path (the result is shown at the end of the path) and then add the probabilities of the paths together. P(win in reg. time) $= 0.76 + 0.05 = 0.81$.

For Strategy “Do Not Foul”

Similarly, we multiply the probabilities along the unique path leading to overtime to find $P(O.T.) = 0.19$.



Strategy: Do Not Foul

Outcome	Probability
Win (W)	0.81
Loss(L)	0
Overtime(O.T.)	0.19

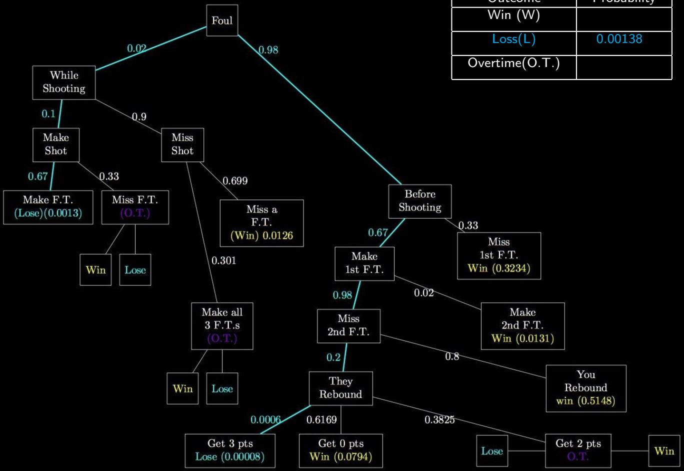
- The probability of a loss if the offense does not foul is 0 (we are assuming that there is only time for one shot)

For Strategy “ Foul”

For the strategy “Foul”, paths leading to losses in regulation time are highlighted in blue below:

Strategy: Foul

Outcome	Probability
Win (W)	
Loss(L)	0.00138
Overtime(O.T.)	

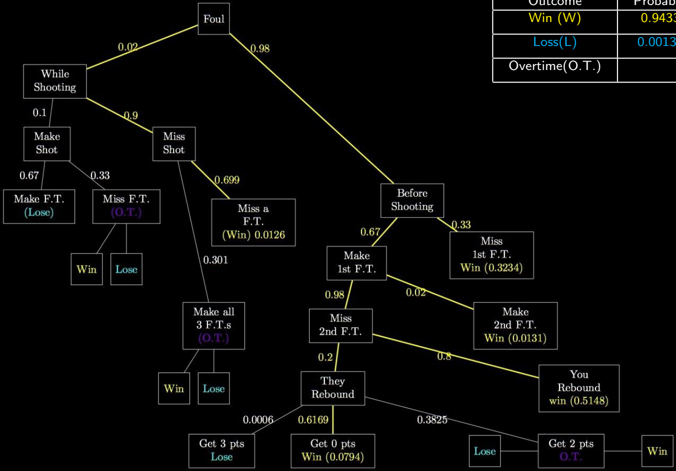


For Strategy “ Foul”

Paths leading to wins in regulation time are highlighted in yellow below:

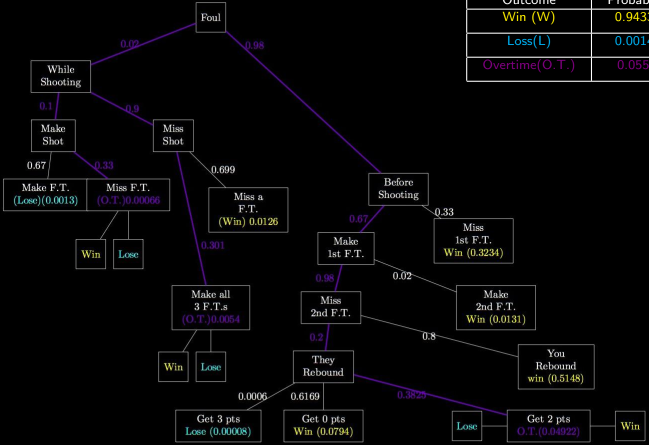
Strategy: Foul

Outcome	Probability
Win (W)	0.9433
Loss(L)	0.00138
Overtime(O.T.)	



For Strategy “ Foul”

Paths corresponding to overtime are highlighted below:

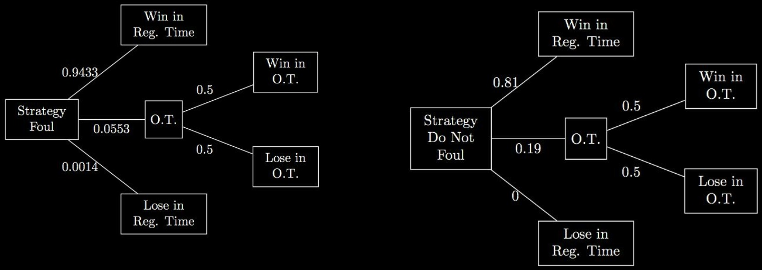


Strategy: Foul

Outcome	Probability
Win (W)	0.9433
Loss(L)	0.0014
Overtime(O.T.)	0.0553

Foul vs. Not Foul

Coach Fenlon does not specify the probability of a win in overtime, but says that the opposing team are more likely to win because of the momentum gained from their success in the final seconds of regular time. Let's assume that the probability of a win in overtime is 0.5 for our team (we're overestimating here). Comparing strategies:



- With the strategy "Foul" the probability of a win is $0.9433 + (0.0553)(0.5) = 0.9709$. With the strategy "Do Not Foul", the probability of a win is $0.81 + (0.19)(0.5) = 0.905$.

Foul vs. Not Foul

Thus we see with the given probabilities that the probability of winning with the strategy "Foul" (0.97) is greater than the probability of winning with the strategy "Do Not Foul" (0.905).

- Given that the team would have a probability less than 0.5 of winning in overtime, the probability of winning with the strategy "Do Not Foul" will always be less than or equal to 0.905.
- The probability of winning with the strategy "Foul" will always be greater than the probability of winning in regulation time with that strategy which is approximately 0.94. So no matter what the probability of a win in overtime is set to be, "Fouling" is always the better strategy.
- Of course the probabilities used by Coach Fenlon were compiled from national statistics for NCAA basketball. These statistics may vary from league to leave or team to team. If coaching, feel free to substitute your best estimates of probabilities from your own situation and redo the calculation to decide on the best strategy.

Alternate Move Games

Sometimes in a game, a player gets to see their opponent's strategy before deciding on their strategy and sometimes both decide on a strategy simultaneously.

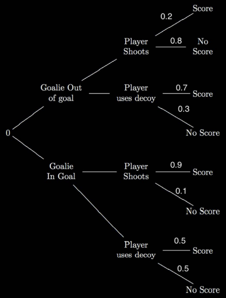
- If the opponent is predictable or telegraphs his/her moves, then we can assume that the players are playing an Alternate move game and a tree diagram should help to choose a strategy.

Example: Alternate Move Games

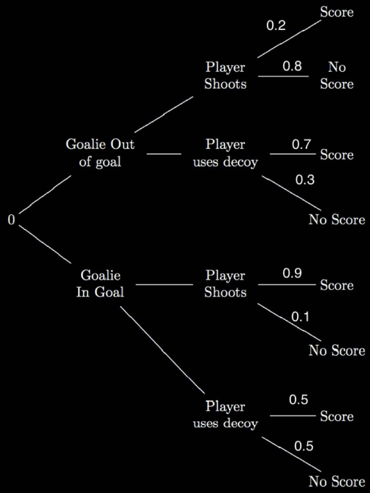
If a hockey player is about to take a shot on goal, the strategy he will use depends on the position of the goalie. If the goalie is out from the goal he may have to use a decoy to get around the goalie, but if the goalie is in the goal (or deep in the crease) then shooting is more likely to be a better strategy than using a decoy.

- Although the model is simplified, the player sitting on the bench can still benefit from pulling out the general principles and making some mental calculations before getting on the ice.

- A player can make an assessment of probabilities and use a tree diagram to represent them as follows:

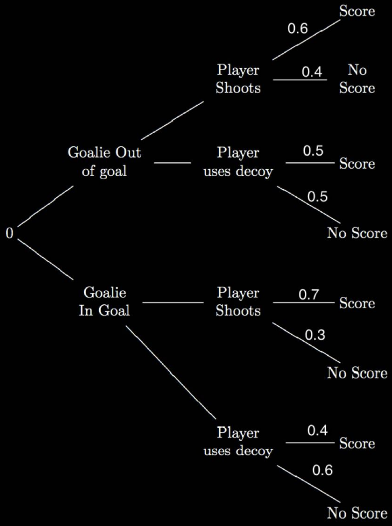


Example: Alternate Move Games



- If the estimated probabilities are as in the diagram on the left, then the player's best strategy can be summarized as follows: "If the goalie is out of the goal the player should use a fake or a decoy and if he is deep in the crease, the player should shoot"

Different Probabilities Change the Strategy



- On the other hand if the estimated probabilities were like those in the diagram to the left for some reason (maybe the goalie does not respond quickly to decoys), the best strategy for the shooter would be: "Always shoot"

Simultaneous Move Games

Note If the goalie is more dynamic and can switch position quickly perhaps sometimes coming out of the crease as the shooter approaches or moving back into the crease, the player should use the analysis for a simultaneous move game which is our next topic

Games

Mathematical game theory was developed as a model of situations of conflict. Such situations and interactions will be called **games** and they have participants who are called players.

- In our course we will focus on simultaneous move games with **exactly two players**.
- Each player receives a payoff depending on the strategies chosen by both players.
- In **zero-sum games** one player's loss is the other player's gain and the payoff to both players for any given scenario adds to zero.

Pay-Off Matrix

Our main tool for analysis is the Pay-Off Matrix:

- Definition The Payoff Matrix for a simultaneous move game is an array whose rows correspond to the strategies of one player (called the Row player) and whose columns correspond to the strategies of the other player (called the Column player).
- Each entry of the array (matrix) is the result, or payoff, obtained when the row player chooses the strategy corresponding to the row associated to the entry and the column player chooses the strategy corresponding to the column associated to the entry.
- This entry is often written as an ordered pair, where the first number represents the payoff for the Row player and the second number represents the payoff for the Column player. We assume that a larger payoff is better for each player.

Example: Pay-Off Matrix

Example: Two fitness chains Fitness Indiana and Get up 'n Go plan to expand by adding one fitness center to one of two neighborhoods neither of which have an existing fitness center.

- The first neighborhood has 5,000 people who would go to a local fitness center and the second neighborhood has 3,000 people who would use a local fitness center.
- If only one fitness center locates in a given neighborhood, that center gains all of the potential customers.
- If the stores locate in the same neighborhood, then 70% of the customers will go to Fitness Indiana, the better known chain.
- Each chain is fully aware of all of these details and must choose the neighborhood for its store without knowing the choice of its competitor.

Example: Pay-Off Matrix

We can summarize the possible choices or strategies of each player and the corresponding payoffs in each possible scenario in a **Payoff matrix** as follows:

		Fitness	**Indiana**
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

- This payoff matrix shows a pair of payoffs for each of the four possible scenarios.
- The first number in each pair is the number of customers that Get up 'n Go will have for that scenario and the second number in each pair shows the number of customers that Fitness Indiana will have in that situation.
- If the stores locate in the same neighborhood, then 70% of the customers will go to Fitness Indiana, the better known chain.
- For example, if both business' locate in the First Neighborhood, Fitness Indiana will get 70% of the customers in that neighborhood (70% of $5000 = 3500$) and Get up 'n Go will have the remaining $5000 - 3500 = 1500$ customers.

Constant-Sum and Zero-Sum Games

A two person zero-sum game is a game where the pair of payoffs for each entry of the payoff matrix sum to 0. This means that one player's gain is equal to the other player's loss on any given play of the game.

- A two person constant-sum game is a game where the pair of payoffs for each entry of the payoff matrix sum to the same constant C. The analysis of these games is the same as that of zero sum games, since subtracting the given constant from the column player's payoffs makes it a zero sum game.
- By Convention the payoff matrix for a two player zero-sum game or a two player constant-sum game, shows the strategies for both players with the payoffs for the row player only as entries. The payoffs for the column player for each situation can be deduced from the row player's payoff.

Example 1: Zero-Sum Game

Example: Rock Paper Scissors In the game of Rock-scissors-paper, the players face each other and simultaneously display their hands in one of the following three shapes: a fist denoting a rock, the forefinger and middle finger extended and spread so as to suggest scissors, or a downward facing palm denoting a sheet of paper. The rock wins over the scissors since it can shatter them, the scissors wins over the paper since they can cut it, and the paper wins over the rock since it can be wrapped around it. The winner collects a penny from the opponent and no money changes hands in the case of a tie.

- The payoff matrix for this game (for the row player) is shown below:

		Colleen		
		Rock	**Paper**	**Scissors**
Roger	**Rock**	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Example 2: Zero-Sum Game

Example: Two Finger Morra Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player's call matches the total, no money changes hands.

- We write each strategy as an ordered pair where the first number denotes the number of finger the player holds up and the second denotes the number that they shout (here the strategy $(1, 2)$ means that the player holds up one finger and shouts 2).
- You should try to fill in the payoffs in the payoff matrix for Ruth before the solutions are revealed in the next slide:

Charlie

		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$
R	$(1,2)$				
u	$(1,3)$				
t	$(2,3)$				
h	$(2,4)$				

Example 2: Zero-Sum Game

Example: Two Finger Morra Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player's call matches the total, no money changes hands.

Charlie

		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$
R	$(1,2)$	0	2	-3	0
u	$(1,3)$	-2	0	0	3
t	$(2,3)$	3	0	0	-4
h	$(2,4)$	0	-3	4	0

- Notice that we do not need to include Charlie's payoff for each scenario, because we can deduce it from the information given. Because this is a zero-sum game, if Ruth wins 2 pennies, Charlie loses two pennies and vice-versa.

Using Expected Value (or Averages) as Payoff's

Sometimes the pay-off for each combination of strategies might be a random variable, in this case we could use the expected payoff or average payoff to calculate the payoff for the row player in a zero sum game.

- Football Run or Pass? [An example from Winston (Mathletics)] In football, the offense selects a play and the defense lines up in a defensive formation.
- We will consider a very simple model of play selection in which the offense and defense simultaneously select their play.
- The offense may choose to run or to pass and the defense may choose a run or a pass defense.
- One can use the average yardage gained or lost in their particular League as payoffs and construct a payoff matrix for this two player zero-sum game.

Using Expected Value (or Averages) as Payoff's

Football Run or Pass? [An example from Winston (Mathletics)] The offense may choose to run or to pass and the defense may choose a run or a pass defense.

- Lets assume that if the offense runs and the defense makes the right call, yards gained average out at a loss of 5 yards for the offense.
- On the other hand if offense runs and defense makes the wrong call, the average gain is 5 yards.
- On a pass, the right defensive call usually results in an incomplete pass averaging out to a zero yard gain for offense and the wrong defensive call leads to a 10 yard gain for offense.
- Set up the payoff matrix for the offensive team for this zero-sum game (try this before I go to the next slide).

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**		
	Pass		

Using Expected Value (or Averages) as Payoff's

Football Run or Pass? [An example from Winston (Mathletics)] The offense may choose to run or to pass and the defense may choose a run or a pass defense.

- Lets assume that if the offense runs and the defense makes the right call, yards gained average out at a loss of 5 yards for the offense.
- On the other hand if offense runs and defense makes the wrong call, the average gain is 5 yards.
- On a pass, the right defensive call usually results in an incomplete pass averaging out to a zero yard gain for offense and the wrong defensive call leads to a 10 yard gain for offense.
- Set up the payoff matrix for the offensive team for this zero-sum game (try this before I go to the next slide).

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**	-5	5
	Pass	10	0

Using Expected Value (or Averages) as Payoff's

Fencing: Sabre In a sabre match in fencing, each fencer can choose to attack straight off the line (A) when the referee gives the signal to begin or the fencer can hold back (H), delaying their attack or taking a defensive position. Bouts are three minutes long, and are fenced to five points. If no fencer reaches five points, then the one with the most points after three minutes wins. Lets assume Rhonda and Cathy are opponents in a saber match.

- Each interaction can result in a point for Rhonda (payoff of 1 for Rhonda), no points for either player(Payoff of 0 for Rhonda) or a point for Cathy (Payoff of -1 for Rhonda).
- No single payoff is guaranteed for any set of strategies, but Rhonda can calculate the probability that each situation will result in one of the payoffs 1, 0 or -1 (for Rhonda) by collecting statistics and observing videos of her opponents.
- To set up the payoff matrix

		Cathy	
		A	H
Rhonda	A		
	H		

, we will calculate the expected payoff for Rhonda for each of the four situations using Rhonda's estimates for the probability distribution for her payoff in each situation.

Using Expected Value (or Averages) as Payoff's

Fencing: Sabre Suppose Rhonda has estimated the probability distributions shown below for her payoffs in each situation, resulting in the following 4 distributions:

R(A), C(A)

Payoff	Probability
1	0.7
0	0.1
-1	0.2

R(A), C(H)

Payoff	Probability
1	0.3
0	0.2
-1	0.5

R(H), C(A)

Payoff	Probability
1	0.3
0	0.1
-1	0.6

R(H), C(H)

Payoff	Probability
1	0.7
0	0.1
-1	0.2

		Cathy	
		A	H
Rhonda	A		
	H		

.

Using Expected Value (or Averages) as Payoff's

Fencing: Sabre We calculate expected payoffs for Rhonda for each situation

R(A), C(A)

Payoff	Probability	XP(X)
1	0.7	.7
0	0.1	0
-1	0.2	-.2
	E(X) =	0.5

R(A), C(H)

Payoff	Probability	XP(X)
1	0.3	0.3
0	0.2	0
-1	0.5	-0.5
	E(X) =	-0.2

R(H), C(A)

Payoff	Probability	XP(X)
1	0.3	.3
0	0.1	0
-1	0.6	- .6
	E(X) =	- 0.3

R(H), C(H)

Payoff	Probability	XP(X)
1	0.7	0.7
0	0.1	0
-1	0.2	-0.2
	E(X) =	0.5

		Cathy	
		A	H
Rhonda	A	0.5	-0.2
	H	- 0.3	0.5

- Note that since Rhonda's gain is always Cathy's loss, Rhonda's expected payoff is always the negative of Cathy's expected payoff.

Using Percentages or probabilities as Payoff's

In a win-loss situation, we can use the probability of a win as the payoff for the row player. This gives us a constant sum game where the probably of a win for both players adds to 1. We can also use percentages as payoffs in a similar way.

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Using Percentages or probabilities as Payoff's

Example (Dutta): Drug Testing Suppose two swimmers, Rogers and Carter, are about to compete in a runoff. Each athlete has the option of using a performance enhancing drug (d) or not using it (n). Lets assume that both competitors have equal abilities and are the two top competitors with no serious competition for first and second place.

- If no drug testing exists we give each a fifty percent chance of winning the race if neither takes the drugs.
- Suppose on the other hand that one takes the drug and one does not, then the one who takes the drug is sure to win.
- Finally if both take the drug, each has a fifty percent chance of winning.
- Using the probability of a win for Rogers to complete the payoff matrix shown below where d denotes the strategy of taking the drug and n denotes the strategy of not taking the drug.

		Carter	
		d	**n**
Rogers	**d**	0.5	1
	n	0	0.5

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Same example, different approach

Example (Dutta): Drug Testing Suppose two swimmers, Rogers and Carter, are about to compete in a runoff. Each athlete has the option of using a performance enhancing drug (d) or not using it (n). Lets assume that both competitors have equal abilities and are the two top competitors with no serious competition for first and second place. If no drug testing exists we give each a fifty percent chance of winning the race if neither takes the drugs. If one takes the drug and one does not, then the one who takes the drug is sure to win. If both take the drug, each has a fifty percent chance of winning.

- We could also make this into a zero sum game by using the expected value as a payoff here with a payoff of $+1$ for the player who wins the race and a payoff of -1 for the player who does not win.
- In this case, the payoff matrix for Rogers looks like:

		Carter	
		d	**n**
Rogers	**d**	0	1
	n	-1	0

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